Hale School

Mathematics Specialist

Test 1 --- Term 1 2017

Complex Numbers

/ 45

Name: ______________________________

Instructions:

- **CAS calculators are NOT allowed**
- **External notes are not allowed**
- **Duration of test: 45 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (Otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

All arguments must be given using principal values.

Question 1 (4 marks: 1, 1, 2)

The following diagram shows a complex number z on the complex plane.

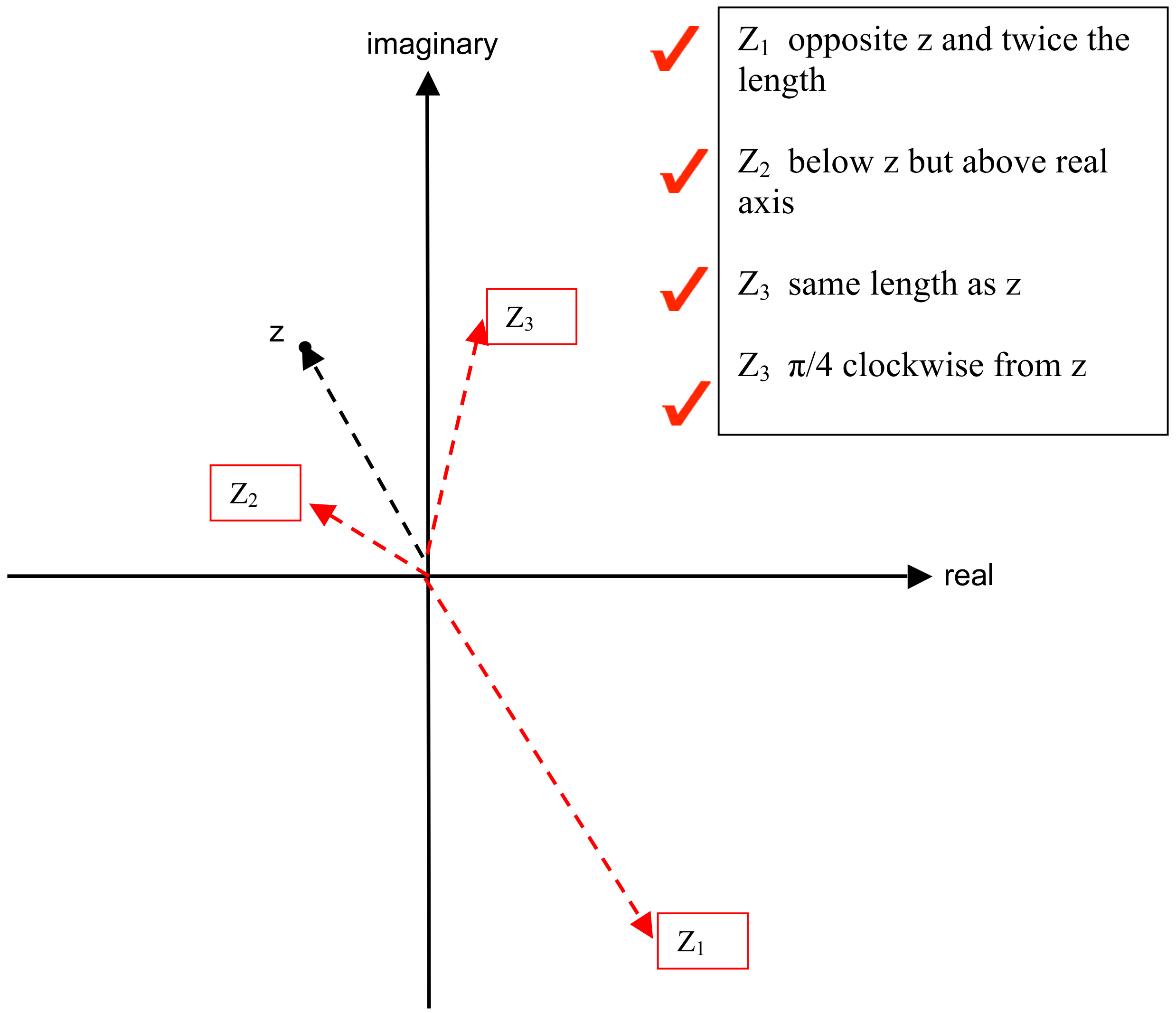
It is known that $|z| = 2$

Locate the following complex numbers. Label your answers clearly.

(a) $z_1 = -2z$ (1 mark)

(b) $z_2 = \bar{z} + 2i$ (1 mark)

(c) $z_3 = \frac{\sqrt{2}\,z}{(1+i)} = \frac{\sqrt{2}\,z}{\sqrt{2}cis(\pi/4)} = zcis(-\pi/4)$ (2 marks)



Question 2 **(6 marks: 2, 4)**

Simplify the following expressions, leaving your answers in rectangular form;

(a) $$\frac{3+4i}{2-3i} = \frac{3+4i}{2-3i} \times \frac{2+3i}{2+3i}$$
$$= \frac{-6+17i}{13}$$

✓ ✓

Multiplies numerator and denominator by conjugate Evaluates the answer

(b) $$\frac{\left(2cis(\pi/8)\right)^4}{\left(\sqrt{2}cis(\pi/4)\right)^5} = \frac{2^4 cis(4\pi/8)}{\left(\sqrt{2}\right)^5 cis(5\pi/4)}$$
$$= \frac{16}{4\sqrt{2}} cis\left(\frac{4\pi-10\pi}{8}\right)$$
$$= 2\sqrt{2}cis\left(\frac{-3\pi}{4}\right)$$
$$= 2\sqrt{2}\left(\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$
$$= -2-2i$$

✓ ✓ ✓ ✓

Uses De Moivre's Thm Modulus $2\sqrt{2}$ Argument -3π/4 Rectangular form

Question 3 **(5 marks)**

Solve the equation $5 - i = z(3 + 2i) + 3\bar{z}$

$$5 - i = z(3 + 2i) + 3\bar{z}$$
$$5 - i = (x + yi)(3 + 2i) + 3(x - yi)$$
$$5 - i = (3x - 2y + 3x) + i(3y + 2x - 3y)$$
$$\text{Real}: 5 = 6x - 2y$$
$$\text{Imag}: -1 = 2x$$
$$x = -1/2, y = -4$$

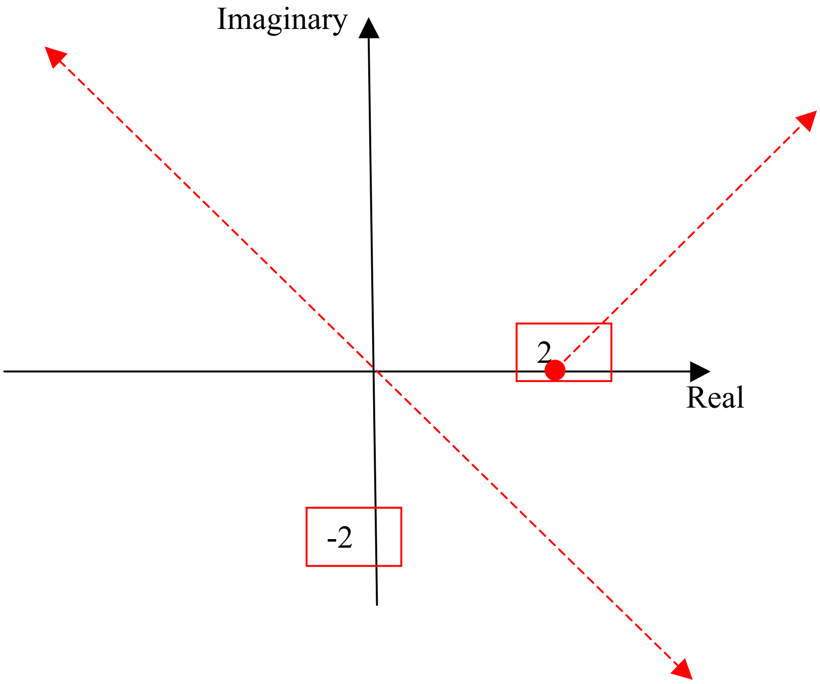
$$\therefore z = -1/2 - 4i$$

✓ Substitutes z = x + yi
✓ Expands correctly
✓ Compares real and imaginary parts
✓ Finds the x and y values
✓ Writes z = -1/2 - 4i

Question 4 **(4 marks: 2, 2)**

On the Argand plane below, sketch the locus of points given by

(a) $|z-2| = |z+2i|$ (b) $|z-2| = |z+2i| - 2\sqrt{2}$



✓ ✓

Full line used for part a)

Correct line shown

✓ ✓

Half line used for part b)

Line starts from Re(z) = 2

Question 5 **(7 marks: 1, 1, 2 and 3)**

Consider the polynomial $f(z) = z^5 + z^3 - 8iz^2 - 8i$.

(a) Show that $(z+i)$ is a factor of $f(z)$ (1 mark)

$$f(-i) = (-i)^5 + (-i)^3 - 8i \times (-i)^2 - 8i$$
$$= -i + i + 8i - 8i = 0$$
$$\therefore (z+i) \quad is\ a\ factor$$

✓ Shows working to justify *f(-i)* = 0

(b) Find another factor of $f(z)$ (1 mark)

$$f(i) = (i)^5 + (i)^3 - 8i \times (i)^2 - 8i = 0$$
$$\therefore (z-i) \quad is\ a\ factor$$

✓ Shows working to justify *f(-i)* = 0

(c) Factorise $f(z)$

$$f(z) = (z+i)(z-i)P(z)$$
$$z^5 + z^3 - 8iz^2 - 8i = (z^2+1)P(z)$$
$$z^5 + z^3 - 8iz^2 - 8i = (z^2+1)(z^3 - 8i)$$

✓ Uses factors of *(z + i)* and *(z - i)*

✓ Finds other factor correctly

(a) Solve the equation $f(z) = 0$, giving answers in polar form. (3 marks)

$$f(z) = (z^2+1)(z^3-8i) = 0$$
$$\Rightarrow \quad z = \pm i \quad or \quad z^3 = 8i = 8cis\left(\frac{\pi}{2}\right)$$
$$\Rightarrow \quad z = i,\ z = -i,\ z = 2cis\left(\frac{\pi}{6}\right), z = 2cis\left(\frac{\pi}{6} + \frac{2\pi}{3}\right), z = 2cis\left(\frac{\pi}{6} - \frac{2\pi}{3}\right)$$
$$\Rightarrow \quad z = cis\left(\frac{\pi}{2}\right), z = cis\left(-\frac{\pi}{2}\right), z = 2cis\left(\frac{\pi}{6}\right), z = 2cis\left(\frac{5\pi}{6}\right), z = 2cis\left(-\frac{\pi}{2}\right)$$

✓ One correct root of 8i

✓ All roots of 8i in polar form

✓ All 5 solutions in polar form

Question 6 (7 marks)

Use De Moivre's Theorem $\left(\cos\theta + i\sin\theta\right)^n = \cos(n\theta) + i\sin(n\theta)$

To prove the trigonometric identity $\sin(3\theta)\cos(2\theta) = 8\sin^5\theta - 10\sin^3\theta + 3\sin\theta$

$$\sin(3\theta) = \text{Im}\left[cis3\theta\right] = \text{Im}\left[(\cos\theta + i\sin\theta)^3\right]$$
$$= 3\cos^2\theta\sin\theta - \sin^3\theta$$
$$\cos(2\theta) = \text{Re}\left[cis2\theta\right] = \text{Re}\left[(\cos\theta + i\sin\theta)^2\right]$$
$$= \cos^2\theta - \sin^2\theta$$

$$\begin{aligned} LHS &= \sin(3\theta)\cos(2\theta) \\ &= \left[3\cos^2\theta\sin\theta - \sin^3\theta\right]\left[\cos^2\theta - \sin^2\theta\right] \\ &= \left[3(1-\sin^2\theta)\sin\theta - \sin^3\theta\right]\left[1 - 2\sin^2\theta\right] \\ &= \left[3\sin\theta - 4\sin^3\theta\right]\left[1 - 2\sin^2\theta\right] \\ &= 8\sin^5\theta - 4\sin^3\theta - 6\sin^3\theta + 3\sin\theta \\ &= 8\sin^5\theta - 10\sin^3\theta + 3\sin\theta \\ &= RHS \end{aligned}$$

Marking criteria
✓ Uses Im(cis(3θ)) and Re(cis(2θ))
✓ Expands $(\text{cis}(\theta))^3$ correctly
✓ Expands $(\text{cis}(\theta))^2$ correctly
✓ Substitutes for sin(3θ) and cos(2θ) correctly
✓ Rewrites using $\cos^2\theta = 1 - \sin^2\theta$
✓ Expands and collects terms
✓ Full and complete proof

Question 7 (5 marks)

It is known that $(z-2+3i)$ is a factor of $f(z)=z^4-4z^3+9z^2+16z-52$.
Use this information to find all the roots of the equation $f(z)=0$

$z-2+3i$ a factor $\Rightarrow$ $z-2-3i$ a factor ✓

$$f(z)=(z-2+3i)(z-2-3i)Q(z)$$
$$=((z-2)^2+9)Q(z)$$ ✓
$$\therefore f(z)=z^4-4z^3+9z^2+16z-52$$
$$=(z^2-4z+13)(z^2-4)$$
$$=(z-2+3i)(z-2-3i)(z-2)(z+2)$$ ✓

The roots are $z=2+3i$, $z=2+3i$, $z=2$, $z=-2$ ✓

Recognises the other factor based on the conjugate

Multiplies two factors to get

$z^2-4z+13$

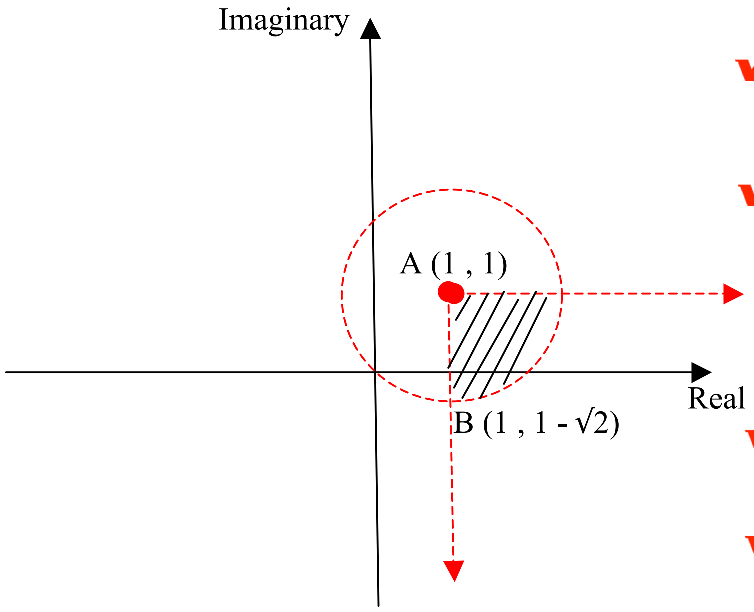
Finds $Q(z)=z^2-4$

States all 4 roots.

Question 8 **(7 marks: 4, 3)**

(a) On the axes below sketch the region of the Argand diagram for which (4 marks)

$$-\frac{\pi}{2} \le \arg(z-1-i) \le 0 \quad \textit{and} \quad |z-1-i| \le \sqrt{2}$$



Draws circle centre (1, 1) radius √2

Draws line segments from the point (1 , 1)

Lines in correct direction

Shades correct region

(b) For the region defined in part (a) above, find the minimum and maximum values for

$\tan\theta \quad where\ \theta = \arg(z)$

Minimum at B where $\tan\theta = \dfrac{1-\sqrt{2}}{1} = 1-\sqrt{2}$

Maximum at A where $\tan\theta = 1$

Recognises A and B as points required.

Finds tanθ = 1 - √2

Finds tanθ = 1